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Classification of the entangled states of $2 \times N \times N$

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Abstract

We develop a novel method in classifying the multipartite entanglement state of $2 \times N \times N$ under stochastic local operation and classical communication. In this method, all inequivalent classes of true entangled state can be sorted directly without knowing the classification information of lower dimension ones for any given dimension N .

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1. Introduction

Entanglement is at the heart of the quantum information theory (QIT) and is now thought of as a physical resource to realize quantum information tasks, such as quantum cryptography [1, 2], superdense coding [3, 4], quantum computation [5], etc. Moreover, the study of entanglement may also improve our knowledge about quantum non-locality [6]. Within entanglement study, the investigation on the classification of multipartite entanglement is of particular importance and interest for QIT. According to QIT, two quantum states can be employed to carry on the same task while they are thought to be equivalent in the meaning of mutually convertible under stochastic local operations and classical communication (SLOCC) [7].

Nevertheless, in practice the classification of multipartite entanglement in high dimension in the Hilbert space is generally mathematically difficult [8]. It was found that the matrix decomposition method [9, 10] tends to be a useful tool as in the two-partite case. A widely adopted philosophy in dealing with this issue is first to classify the state in the lower dimension (or less partite) and then extend to the higher dimension [11, 12] (or more partite [13, 14]) cases in an inductive way. However, nontrivial aspect emerges as the dimension increases, i.e. some non-local parameters may nest in the entangled states [15, 16]. In recent years, investigations on the classification of $2 \times M \times N$ states were performed [11, 12], where M and N are

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dimensions of two partites in three-partite entangled states. Based on the ‘range criterion’, an iterated method was introduced to determine all classes of true entangled states of the $2 \times M \times N$ system in [11, 12]. In this scenario the entanglement classes of high-dimensional states can be obtained through the low-dimensional ones. That is, first generate all the possible entanglement classes under invertible local operator (ILO) by the classification information of lower dimensional ones, then use the ‘range criterion’ to find out the inequivalent classes of true entanglement among all the possible entangled classes, which tends to be a formidable task with the increase of dimensions. The main trait of this scenario is that the lower dimensional entanglement classes are prerequisite for the follow-up classification. As mentioned in [12] the classification of the entangled state of $2 \times M \times N$ becomes more subtle when $M = N$. In this case the permutations of the two N -dimensional partites may be sorted into different classes.

In this work, we present a straightforward method in fully classifying the entanglement states in $2 \times N \times N$ configuration. The asymmetry of the two N -dimensional partites shows up in one of the classes. We develop a cubic grid form for the quantum state, in which the entangled classes can be visualized explicitly. This may give an instructive insight on the entanglement classes of four or more partites which also have non-local parameters [7, 14].

The paper is arranged as follows: after the introduction section, we represent the entangled state in a general form in section 2, by which the true entangled state of $2 \times N \times N$ can be expressed in a matrix pair. With the definitions given in section 2, the true entangled state of $2 \times N \times N$ can be fully classified according to the theorems given in section 3. In section 4, two examples on how to employ the novel classification method are presented. The last section contains a brief summary.

2. Representation of the entangled states of $2 \times N \times N$

An arbitrary two-partite state in dimensions of $M \times N$ can be expressed in the following form:

$$\begin{aligned} |\Psi_{M \times N}\rangle &= \gamma_{11}|11\rangle + \gamma_{12}|12\rangle + \cdots + \gamma_{1N}|1N\rangle \\ &\quad + \gamma_{21}|21\rangle + \gamma_{22}|22\rangle + \cdots + \gamma_{2N}|2N\rangle \\ &\quad + \vdots \\ &\quad \gamma_{M1}|M1\rangle + \gamma_{M2}|M2\rangle + \cdots + \gamma_{MN}|MN\rangle, \end{aligned} \quad (1)$$

where $\gamma_{ij} \in \mathbb{C}$ are a series of complex numbers. Equation (1) can be further expressed in a more compact form

$$\begin{aligned} |\Psi_{M \times N}\rangle &= (|1\rangle, |2\rangle, \dots, |M\rangle) \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1N} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{M1} & \gamma_{M2} & \cdots & \gamma_{MN} \end{pmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \\ \vdots \\ |N\rangle \end{pmatrix} \\ &\equiv \psi_1^T \Gamma_{\{i,j\}} \psi_2 = Tr[\Gamma_{\{i,j\}} \psi_2 \otimes \psi_1^T]. \end{aligned} \quad (2)$$

Here, $\Gamma_{\{i,j\}}$ denotes the $M \times N$ complex matrix, which can also be treated as a tensor of rank 2, and \otimes is the symbol of direct product. Obviously, the feature of an $M \times N$ pure state is characterized by the rank-2 tensor $\Gamma_{\{i,j\}}$. Similarly, the state of $2 \times N \times N$ may be expressed in a traced form

$$|\Psi_{2 \times N \times N}\rangle = Tr[\Gamma_{\{i,j,k\}} \psi_2 \otimes \psi_1^T \otimes \psi_0^T], \quad (3)$$

where ψ_0 is a two-dimensional vector and $\psi_{1,2}$ are N -dimensional vectors, representing the constituent states in Hilbert space. The $2 \times N \times N$ matrix $\Gamma_{\{i,j,k\}}$, which can also be taken as a rank-3 tensor, reads

$$\Gamma_{\{i,j,k\}} = \begin{pmatrix} \gamma_{111} & \gamma_{112} & \cdots & \gamma_{11N} \\ \gamma_{121} & \gamma_{122} & \cdots & \gamma_{12N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{1N1} & \gamma_{1N2} & \cdots & \gamma_{1NN} \\ \gamma_{211} & \gamma_{212} & \cdots & \gamma_{21N} \\ \gamma_{221} & \gamma_{222} & \cdots & \gamma_{22N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{2N1} & \gamma_{2N2} & \cdots & \gamma_{2NN} \end{pmatrix} = \begin{pmatrix} \Gamma_{\{1,l,m\}} \\ \Gamma_{\{2,l,m\}} \end{pmatrix} \equiv \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}. \quad (4)$$

Here, $\Gamma_{\{1,l,m\}}$ and $\Gamma_{\{2,l,m\}}$ are in fact tensors of rank 2, which are represented by $N \times N$ complex matrices $\Gamma_{1,2}$. Γ_1 and Γ_2 stand for the upper and lower N -line blocks of the matrix $\Gamma_{\{i,j,k\}}$, respectively.

From (3) and (4) we know that the entanglement information of the state $2 \times N \times N$ rests in the matrix pair (Γ_1, Γ_2) . Therefore, with the aim of classification we can specify the typical entangled state by a ‘matrix vector’, that is

$$|\Psi_{2 \times N \times N}\rangle \doteq \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}, \quad (5)$$

where the symbol \doteq stands for ‘is represented by’.

Generally speaking, two states are said to be SLOCC equivalent if they are connected by ILOs [7]. For instance, in the case of bipartite entanglement, suppose the two parties are transformed under two invertible local operators P and Q , i.e. $\psi'_1 = P^T \psi_1$, $\psi'_2 = Q \psi_2$; then from equation (2) a SLOCC equivalent state to this bipartite entangled state reads

$$\begin{aligned} |\Psi'_{M \times N}\rangle &= \psi_1'^T \Gamma_{\{i,j\}} \psi_2' \\ &= \text{Tr}[P \Gamma_{\{i,j\}} Q \psi_2 \otimes \psi_1^T] \\ &= \psi_1'^T \Gamma'_{\{i,j\}} \psi_2. \end{aligned} \quad (6)$$

From the above expression, we see that two SLOCC equivalent states are in fact connected only by the transformation of the matrix $\Gamma_{\{i,j\}}$ in equation (2) like

$$\Gamma'_{\{i,j\}} = P \Gamma_{\{i,j\}} Q. \quad (7)$$

Similarly, two SLOCC equivalent $2 \times N \times N$ states $|\Psi'_{2 \times N \times N}\rangle$ and $|\Psi_{2 \times N \times N}\rangle$ are also connected by the transformation of the matrix $\Gamma_{\{i,j,k\}}$ in equation (4), i.e.

$$\begin{aligned} |\Psi'_{2 \times N \times N}\rangle &= \text{Tr}[\Gamma_{\{i,j,k\}} \psi_2' \otimes \psi_1'^T \otimes \psi_0^T] \\ &= \text{Tr}[T \otimes P \Gamma_{\{i,j,k\}} Q \psi_2 \otimes \psi_1^T \otimes \psi_0^T] \\ &= \text{Tr}[\Gamma'_{\{i,j,k\}} \psi_2 \otimes \psi_1^T \otimes \psi_0^T], \end{aligned} \quad (8)$$

where

$$\Gamma'_{\{i,j,k\}} = \begin{pmatrix} \Gamma'_{\{1,j,k\}} \\ \Gamma'_{\{2,j,k\}} \end{pmatrix} = T \begin{pmatrix} P \Gamma_1 Q \\ P \Gamma_2 Q \end{pmatrix}. \quad (9)$$

Here, T is an arbitrary invertible 2×2 matrix which acts on ψ_0 ; P and Q are two invertible $N \times N$ matrices acting on ψ_1 and ψ_2 , respectively. The transformation of the first partite T reads

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}. \quad (10)$$

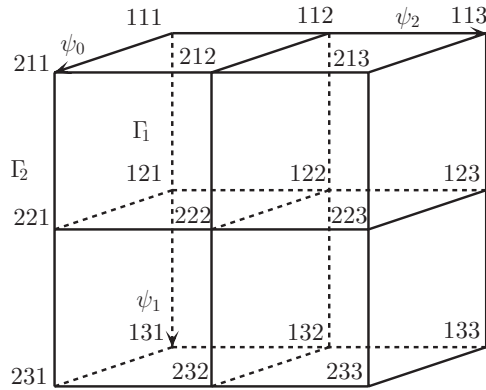


Figure 1. The pictorial description of the $2 \times 3 \times 3$ state, where each node corresponds to a base vector. Assigning a coefficient to the base vector, we then obtain the corresponding matrix element of Γ_1 or Γ_2 .

For brevity, as in (5), equation (8) can be formulated as

$$|\Psi'_{2 \times N \times N}\rangle = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} P\Gamma_1 Q \\ P\Gamma_2 Q \end{pmatrix}, \tag{11}$$

where ψ 's are suppressed.

To give a pictorial description of the quantum state, we take a $2 \times 3 \times 3$ case as an example, see figure 1. The matrices Γ_1 and Γ_2 are placed in parallel in rear and front of the cubic grid, respectively. Of the cubic grid, each node corresponds to an element in the matrix pair (Γ_1, Γ_2) in equation (5).

In the matrix algebra, every ILO which acts on a given matrix can be decomposed as a series of products of elementary operations on the matrix, and there exist three such elementary operations [17]. Therefore, the matrices T , P and Q in equation (8), which connect the two equivalent wavefunctions, can be decomposed as such a sequence of elementary operations. In the pictorial language, here the three types of elementary operation correspond to three types of manipulation upon the cubic grid: *type 1*, interchange of two surfaces; *type 2*, multiplication of one surface by a nonzero scalar; *type 3*, addition of a scalar multiple of one surface to another surface. Specifically, T is responsible for the elementary operations between front and rear; P for upper and lower, Q for left and right surfaces, respectively.

According to the common definition [7], a true $2 \times N \times N$ entangled state requires the following conditions:

$$r(\rho_{\psi_0}) = 2, \quad r(\rho_{\psi_1}) = r(\rho_{\psi_2}) = N, \tag{12}$$

to be true, where $\rho_i = \text{Tr}_{jk}(\rho_{ijk})$ is the reduced density matrix. Hereafter, we denote by r the rank of matrix. In quantum mechanics, to each state there corresponds a unique state operator, the density matrix. In the representation of a matrix pair the density matrix (elements) can be expressed as

$$\rho_{\psi_0, \psi_1, \psi_2} = \Gamma_{ijk} \Gamma_{i'j'k'}^*, \tag{13}$$

where $i, i' = 1, 2; j, j' = 1, 2, \dots, N; k, k' = 1, 2, \dots, N$. The corresponding reduced density matrix is (taking ψ_2 as an example)

$$\begin{aligned}
 \rho_{\psi_2} &= \text{Tr}_{\psi_0, \psi_1}(\rho_{\psi_0, \psi_1, \psi_2}) \\
 &= \sum_{ij} \Gamma_{ijk} \Gamma_{ijk}^* \\
 &= \sum_i (\Gamma_i^\dagger \Gamma_i)_{kk}.
 \end{aligned} \tag{14}$$

If $\text{Det}(\rho_{\psi_2}) \neq 0$, then we know $r(\rho_{\psi_2}) = N$.

3. Classification of the tripartite entangled state $2 \times N \times N$

With the above preparation, we can now proceed to classify the $2 \times N \times N$ state. Generically, the whole space of the state (Γ_1, Γ_2) can be partitioned into numbers of inequivalent sets by different l and n :

$$(\Gamma_1, \Gamma_2) = \{C_{n,l}\}, \tag{15}$$

$$C_{n,l} = \{(\Gamma_1, \Gamma_2) | r_{\max}(\alpha_1 \Gamma_1 + \beta_1 \Gamma_2) = n, r_{\min}(\alpha_2 \Gamma_1 + \beta_2 \Gamma_2) = l\}, \tag{16}$$

where $\alpha_i, \beta_i \in \mathbb{C}$ and $|\alpha_i| + |\beta_i| \neq 0$; $l \in [0, n]$ and $n \in [0, N]$; r_{\max} and r_{\min} are the maximum and minimum ranks of matrices for all possible values of α_i and β_i . From the definition of (16), there is no common element in different sets, i.e. $C_{n,l} \cap C_{m,k} = C_{n,l} \delta_{m,n} \delta_{l,k}$. Obviously, every (entangled) state (Γ_1, Γ_2) must lie in one of the subspaces of the set $\{C_{n,l}\}$ with certain n and l , which in principle can be determined via the transformation of equation (11), since one can always classify a set by certain rules voluntarily. Here, the criteria $r_{\max}(\alpha_1 \Gamma_1 + \beta_1 \Gamma_2) = n$ and $r_{\min}(\alpha_2 \Gamma_1 + \beta_2 \Gamma_2) = l$ attribute to the group $SL(2, \mathbb{C})$ transformation T in (9). Note that in case $\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$ is non-invertible, when $r_{\max}, r_{\min}, \text{Rank}(\Gamma_1)$ and $\text{Rank}(\Gamma_2)$ are all equal in magnitude, its function can be fulfilled by a unit matrix.

Suppose $(\Gamma_1, \Gamma_2) \in C_{n,l}, (\bar{\Gamma}_1, \bar{\Gamma}_2) \in C_{\bar{n}, \bar{l}}$ and

$$\text{Rank} \left[O_A \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \right] = \text{Rank} \left[\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \right] = \text{Rank} \begin{pmatrix} \Gamma'_1 \\ \Gamma'_2 \end{pmatrix} = \begin{pmatrix} n \\ l \end{pmatrix}, \tag{17}$$

$$\text{Rank} \left[\bar{O}_A \begin{pmatrix} \bar{\Gamma}_1 \\ \bar{\Gamma}_2 \end{pmatrix} \right] = \text{Rank} \left[\begin{pmatrix} \bar{\alpha}_1 & \bar{\beta}_1 \\ \bar{\alpha}_2 & \bar{\beta}_2 \end{pmatrix} \begin{pmatrix} \bar{\Gamma}_1 \\ \bar{\Gamma}_2 \end{pmatrix} \right] = \text{Rank} \begin{pmatrix} \bar{\Gamma}'_1 \\ \bar{\Gamma}'_2 \end{pmatrix} = \begin{pmatrix} \bar{n} \\ \bar{l} \end{pmatrix}, \tag{18}$$

where O_A and \bar{O}_A are invertible operators; the ‘Rank’ denotes the rank operation on Γ matrices in upper and lower blocks separately, and no invertible matrix O_I will exist, which enables

$$O_I \begin{pmatrix} \bar{\Gamma}_1 \\ \bar{\Gamma}_2 \end{pmatrix} = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \tag{19}$$

in the case $n \neq \bar{n}$ or $l \neq \bar{l}$, i.e. (Γ_1, Γ_2) and $(\bar{\Gamma}_1, \bar{\Gamma}_2)$ are SLOCC inequivalent. If the operator O_I exists, substituting (19) into (17) we may have

$$\text{Rank} \left[O_A O_I \begin{pmatrix} \bar{\Gamma}_1 \\ \bar{\Gamma}_2 \end{pmatrix} \right] = \begin{pmatrix} n \\ l \end{pmatrix}, \tag{20}$$

and from (18) one knows that $n \leq \bar{n}$ and $l \geq \bar{l}$. Similarly, since O_I is an invertible operator (matrix), one may also get $\bar{n} \leq n$ and $\bar{l} \geq l$, and hence $\bar{n} = n$ and $\bar{l} = l$. From the above arguments, two states, the matrix pairs (Γ_1, Γ_2) and $(\bar{\Gamma}_1, \bar{\Gamma}_2)$, connected via an invertible operator belong to the same subset $C_{n,l}$.

In all, the entangled classes in the set $\{C_{n,l}\}$ with different n and l are SLOCC inequivalent, and the question of performing a complete classification on entangled states now turns to how to classify the entangled states in the subset $C_{n,l}$.

3.1. Classification on the set $C_{n,l}$ with $n = N$

From the definition of $C_{N,l}$ we know that if $(\Gamma_1, \Gamma_2) \in C_{N,l}$, then there exists an invertible operator T which enables

$$T \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} \Gamma'_1 \\ \Gamma'_2 \end{pmatrix}, \quad (21)$$

where Γ'_1 has the maximum rank N and Γ'_2 has the minimum rank l . According to the matrix algebra, in principle one can find invertible operators P, Q and S which further transform (Γ'_1, Γ'_2) in the following form:

$$SP \otimes QS^{-1} \begin{bmatrix} \Gamma'_1 \\ \Gamma'_2 \end{bmatrix} \equiv \begin{bmatrix} SP\Gamma'_1QS^{-1} \\ SP\Gamma'_2QS^{-1} \end{bmatrix} = \begin{bmatrix} E \\ J \end{bmatrix}. \quad (22)$$

Here, $r(J) = r_{\min}(\alpha_2\Gamma_1 + \beta_2\Gamma_2)$ with J a matrix in the Jordan canonical form. A typical form of J reads

$$J = \begin{pmatrix} J_{n_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{n_2}(\lambda_2) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & J_{n_k}(\lambda_k) \end{pmatrix}, \quad (23)$$

in which $J_{n_i}(\lambda_i)$ is an $n_i \times n_i$ matrix which has the following form:

$$J_{n_i}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & & 0 & 0 \\ 0 & 0 & \lambda_i & & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}. \quad (24)$$

In all, for every $(\Gamma_1, \Gamma_2) \in C_{N,l}$, there exists an ILO transformation, like

$$\begin{bmatrix} E \\ J \end{bmatrix} = T \otimes P \otimes Q \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}. \quad (25)$$

Provided $r(J) = N$, we know that the rank of the matrix $J' \equiv (J - \lambda_i E)$, with λ_i being any eigenvalue of J , must be less than that of J 's. This conclusion contradicts with the proviso of J having the minimum rank, since J' and J are correlated through an invertible operator, let us say T' ,

$$\begin{bmatrix} E \\ J' \end{bmatrix} = T' \begin{bmatrix} E \\ J \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ -\lambda_i & 1 \end{pmatrix} \begin{bmatrix} E \\ J \end{bmatrix}. \quad (26)$$

From the above arguments, one observes that the rank of J is less than N , i.e. $l \leq N - 1$. In the special case of $N = 2$, this observation agrees with the proposition given in [13, 18]. From equation (25) $c_{N,l} = (E, J)$ is equivalent to $C_{N,l}$ under the joint invertible transformations of T, P and Q , which means the classification on $C_{N,l}$ can be simply performed on $c_{N,l}$.

From equation (14) one can find that for the quantum state (matrix pair) in $c_{N,l}$:

$$\text{Det}(\rho_{\psi_j}) = \prod_i \left[\sum_{m=0}^{n_i} \frac{(1 + |\lambda_i|^2)^m}{(n_i - m)!} f_{m+1}^{(n_i-m)}(x) \Big|_{x=0} \right] \neq 0 \tag{27}$$

with $f_n^{(m)}(x) = \frac{d^m}{dx^m} \left[\frac{1-x}{1-x-x^2} \right]^n$. Here, $j = \psi_1, \psi_2$ and n_i, λ_i are defined in equation (24). This tells that $r(\rho_{\psi_1}) = r(\rho_{\psi_2}) = N$. When $l \neq 0$, we readily have $r(\rho_{\psi_0}) = 2$. This means that the state in $c_{N,l}$ is the true entangled $2 \times N \times N$ state, while $l = 0$. Otherwise it will not be a true entangled $2 \times N \times N$ state, which is beyond our consideration.

Theorem 1. $\forall (E, J) \in c_{N,l}$, the set $c_{N,l}$ is of the classification of $C_{N,l}$ under SLOCC:

- (i) if two states in $C_{N,l}$ are SLOCC equivalent, then they can be transformed into the same matrix vector (E, J) ;
- (ii) matrix vector (E, J) is unique in $c_{N,l}$ up to a trivial transformation, that is if (E, J') is SLOCC equivalent with (E, J) , then $(E, J') = (E, J + \lambda E)$ with λ being an arbitrary complex number.

Proof.

- (i) Suppose there exists the transformation

$$\begin{bmatrix} \Gamma'_1 \\ \Gamma'_2 \end{bmatrix} = T' \otimes P' \otimes Q' \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}, \tag{28}$$

according to equation (25)

$$\begin{bmatrix} E \\ J \end{bmatrix} = T \cdot T'^{-1} \otimes P \cdot P'^{-1} \otimes Q \cdot Q'^{-1} \begin{bmatrix} \Gamma'_1 \\ \Gamma'_2 \end{bmatrix}. \tag{29}$$

- (ii) Suppose

$$\begin{bmatrix} E \\ J' \end{bmatrix} = T' \otimes P' \otimes Q' \begin{bmatrix} E \\ J \end{bmatrix}, \tag{30}$$

as noted beneath equation (23), we have $l \leq N - 1$, and it tells that there are no zero elements in the pivot of T' . Then T' can be decomposed as lower and upper triangular (LU) forms [19]

$$\begin{bmatrix} t'_{11} & t'_{12} \\ t'_{21} & t'_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}, \tag{31}$$

where $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ and both matrices on the right-hand side are nonsingular. Now equation (30) becomes

$$\begin{bmatrix} E \\ J' \end{bmatrix} = O_1 O_2 \begin{bmatrix} E \\ J \end{bmatrix} \tag{32}$$

with

$$O_1 = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}, \quad O_2 = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \otimes P' \otimes Q'. \tag{33}$$

Here, according to the definition of operator Q in equations (11) and (22), Q' is applied to the matrix vector from the right-hand side. Thus, the operator O_2 acts on (E, J) in the following way:

$$O_2 \begin{bmatrix} E \\ J \end{bmatrix} = P' \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} E \\ J \end{bmatrix} Q', \tag{34}$$

where $J = J(\lambda_i) = \oplus_i J_{n_i}(\lambda_i)$. Equation (32) now gives two independent equations:

$$E = P'(\alpha E + \beta J)Q', \tag{35}$$

$$J' = \lambda E + \gamma P' J Q'. \tag{36}$$

The first one can be reformulated into a different form

$$P' J(\alpha + \beta \lambda_i) Q' = E, \tag{37}$$

and from equations (35) and (37) we can further get

$$\begin{aligned} \frac{\beta}{\alpha} P' J Q' &= \frac{1}{\alpha} E - P' Q' \\ &= \frac{1}{\alpha} E - Q'^{-1} J^{-1}(\alpha + \beta \lambda_i) Q' \\ &= Q'^{-1} M^{-1} \left(\frac{1}{\alpha} E - J \left(\frac{1}{\alpha + \beta \lambda_i} \right) \right) M Q' \\ &= Q'^{-1} M^{-1} J \left(\frac{\beta \lambda_i}{\alpha(\alpha + \beta \lambda_i)} \right) M Q'. \end{aligned} \tag{38}$$

Here, M is an invertible matrix, and theorem (6.2.25) in [20] is employed. Thus,

$$\begin{aligned} \gamma P' J Q' &= Q'^{-1} M^{-1} J \left(\frac{\gamma \lambda_i}{\alpha + \beta \lambda_i} \right) M Q' \\ &= S J \left(\frac{\gamma \lambda_i}{\alpha + \beta \lambda_i} \right) S^{-1} \end{aligned} \tag{39}$$

with $S = Q'^{-1} M^{-1}$. Therefore, from (34)–(39) we get

$$O_2 \begin{bmatrix} E \\ J \end{bmatrix} = \begin{bmatrix} E \\ S \oplus_i J_{n_i} \left(\frac{\gamma \lambda_i}{\alpha + \beta \lambda_i} \right) S^{-1} \end{bmatrix}, \tag{40}$$

and hence

$$\begin{bmatrix} E \\ J' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} E \\ J \end{bmatrix}. \tag{41}$$

□

3.2. Classification on the set $C_{n,l}$ with $n = N - 1$

Having classified the $C_{N,l}$, now we proceed to the $C_{N-1,l}$ case. For every $(\Gamma_1, \Gamma_2) \in C_{N-1,l}$, we can find an ILO T which implements the following transformation:

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} \Gamma'_1 \\ \Gamma'_2 \end{bmatrix}, \tag{42}$$

where $r(\Gamma'_1) = N - 1$ and $r(\Gamma'_2) = l$. Then in principle one can find ILOs P_1 and Q_1 , which transform the matrix vector (Γ'_1, Γ'_2) into the form (Λ, Γ''_2) where Λ is an $N \times N$ diagonal matrix with $N - 1$ nonzero elements of 1 and one zero, and Γ''_2 is another $N \times N$ matrix in a specific form. To be more explicit, taking $N = 6$ as an example (but the procedure is N independent) the above-mentioned procedure tells

$$\Lambda = P\Gamma'_1 Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{43}$$

$$\Gamma''_2 = P\Gamma'_2 Q = \left(\begin{array}{ccc|ccc} \times & \times & \times & 0 & \times & 0 \\ \times & \times & \times & 0 & \times & 0 \\ \times & \times & \times & 0 & \times & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right) = \begin{pmatrix} A & c \\ r & B_3 \end{pmatrix}, \tag{44}$$

where A , B_3 , c and r are submatrices of Γ''_2 partitioned by vertical and horizontal lines; \times means no constraint on the corresponding element. Note that for the case of an $N \times N$ matrix, Γ''_2 can also be partitioned into a similar form as (44), with the lower-right block to be still B_3 (see appendix A for details).

In general, there are four different choices for c and r , i.e. (1) $c = 0$, $r = 0$; (2) $c \neq 0$, $r = 0$; (3) $c = 0$, $r \neq 0$; (4) $c \neq 0$, $r \neq 0$. In these cases Γ''_2 can be further simplified through a series of elementary operations, e.g. denoted by P_k and Q_k , which enables

$$\begin{bmatrix} \Lambda \\ \Gamma''_2 \end{bmatrix} = \begin{bmatrix} P_k \Lambda Q_k \\ P_k \Gamma''_2 Q_k \end{bmatrix}. \tag{45}$$

Here, the superscripts c and r stand for different choices mentioned above, and $\Gamma''_2{}^{cr}$ read

$$\begin{aligned} \Gamma''_2{}^{00} &= \begin{pmatrix} \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, & \Gamma''_2{}^{10} &= \left(\begin{array}{ccc|ccc} \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right), \\ \Gamma''_2{}^{01} &= \left(\begin{array}{ccc|ccc} \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right), & \Gamma''_2{}^{11} &= \left(\begin{array}{ccc|ccc} \times & 0 & \times & 0 & 0 & 0 \\ \times & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right). \end{aligned} \tag{46}$$

In the case of $\Gamma''_2{}^{00}$, it has already been in the form of $\begin{pmatrix} A & 0 \\ 0 & B_3 \end{pmatrix}$. In the other three cases we can repartition the submatrices like

$$\Gamma''_2{}^{10} = \left(\begin{array}{ccc|ccc} \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right) = \begin{pmatrix} A & c \\ 0 & B_4 \end{pmatrix}, \tag{47}$$

$$\Gamma_2''^{01} = \left(\begin{array}{cc|cccc} \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right) = \begin{pmatrix} A & 0 \\ r & B_4 \end{pmatrix}, \quad (48)$$

$$\Gamma_2''^{11} = \left(\begin{array}{c|ccccc} \times & 0 & \times & 0 & 0 & 0 \\ \times & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right) = \begin{pmatrix} A & c \\ r & B_5 \end{pmatrix}, \quad (49)$$

where n in B_n means the dimension of the matrix. This procedure results in the enlargement of the blocks B_n , the shrink of blocks A and the emergence of new types of off-diagonal blocks c and r .

By the same procedure, one can further simplify, enlarge B_n and shrink A , the $\Gamma_2''^{cr}$ of the forms (47)–(49). Finally the Γ_2'' in (44) may arrive at the form of

$$\Gamma_2'' \sim \begin{pmatrix} A & 0 \\ 0 & B_n \end{pmatrix} = \begin{pmatrix} DJD^{-1} & 0 \\ 0 & B_n \end{pmatrix} \quad (50)$$

with different kinds of B_n 's, correspondingly. Here, J is the Jordan canonical form of A , and D is an invertible matrix. Every Γ_2'' matrix has its own specific form of B_n block. Note that B_n may be recursively enlarged to be the whole matrix of Γ_2'' , i.e. $n = N$. In all, for every $(\Gamma_1, \Gamma_2) \in C_{N-1,l}$, there exists an ILO transformation, like

$$\begin{bmatrix} \Lambda \\ \Gamma \end{bmatrix} = T \otimes P \otimes Q \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}. \quad (51)$$

Here, $\Gamma = \begin{pmatrix} J & 0 \\ 0 & B_n \end{pmatrix}$, $P = \prod_i P_i$ and $Q = \prod_i Q_i$, where P_i, Q_i stand for those operators in equations (44), (45) and (50). From equation (51) the subset $c_{N-1,l}$, defined as

$$c_{N-1,l} = \left\{ (\Lambda, \Gamma) | r(\Gamma) = l; \Gamma = \begin{pmatrix} J & 0 \\ 0 & B_n \end{pmatrix}; (\Lambda, \Gamma) \in C_{N-1,l} \right\}, \quad (52)$$

is equivalent to $C_{N-1,l}$ under the joint invertible transformations of T, P and Q , which means that the classification on $C_{N-1,l}$ can be simply performed on $c_{N-1,l}$.

Similarly as (27) we find

$$\text{Det}(\rho_j) = \prod_i \left[\sum_{m=0}^{n_i} \frac{(1 + |\lambda_i|^2)^m}{(n_i - m)!} f_{m+1}^{(n_i-m)}(x) \Big|_{x=0} \right] \cdot 2^{r(B_n)-1} \neq 0, \quad (53)$$

where $j = \psi_1, \psi_2$, and n_i, λ_i are defined as the Jordan blocks in equation (50), and conclude that all the states in $c_{N-1,l}$ are truly entangled in the form of $2 \times N \times N$.

Theorem 2. $\forall (\Lambda, \Gamma) \in c_{N-1,l}$, the set $c_{N-1,l}$ is of the classification of $C_{N-1,l}$ under SLOCC:

- (i) suppose two states in $C_{N-1,l}$ are SLOCC equivalent, and they can then be transformed into the same matrix vector (Λ, Γ) ;
- (ii) the matrix vector in $c_{N-1,l}$ is unique up to a nonzero classification irrelevant parameter, i.e. provided (Λ, Γ') is SLOCC equivalent with (Λ, Γ) ; then $(\Lambda, \Gamma') = (\Lambda, \kappa\Gamma)$ in the sense of J' equalling to J as in theorem 1 while B'_n being exactly the same as B_n .

Proof.

- (i) According to the property of transitivity in SLOCC transformation, this proposition should be true.
- (ii) Suppose

$$\begin{pmatrix} \Lambda \\ \Gamma' \end{pmatrix} = T' \otimes P' \otimes Q' \begin{pmatrix} \Lambda \\ \Gamma \end{pmatrix}; \tag{54}$$

we first demonstrate that the three ILO transformations T', P', Q' can always be fulfilled through two ILO transformations P'', Q'' , that is

$$\begin{aligned} T' \otimes P' \otimes Q' \begin{pmatrix} \Lambda \\ \Gamma \end{pmatrix} &= P'' \otimes Q'' \begin{pmatrix} \Lambda \\ \kappa \Gamma \end{pmatrix} \\ &= \begin{pmatrix} P'' \Lambda Q'' \\ P'' \kappa \Gamma Q'' \end{pmatrix}. \end{aligned} \tag{55}$$

According to the definition of $c_{N-1,l}$ we can write (Λ, Γ) in the form of direct sums

$$\begin{pmatrix} \Lambda \\ \Gamma \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} E & 0 \\ 0 & \Lambda' \end{pmatrix} \\ \begin{pmatrix} J & 0 \\ 0 & B_n \end{pmatrix} \end{pmatrix}, \tag{56}$$

where

$$\Lambda' = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{57}$$

is a square matrix, and its dimension equals to that of B_n . The transformation T'

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \begin{pmatrix} \Lambda \\ \Gamma \end{pmatrix} \tag{58}$$

can be decomposed into the following form:

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \begin{pmatrix} E \\ J \end{pmatrix}, \tag{59}$$

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \begin{pmatrix} \Lambda' \\ B_n \end{pmatrix}, \tag{60}$$

according to the nature of direct sum.

For equation (59), from the proof of theorem 1 one can find the P_J, Q_J

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \begin{pmatrix} P_J E Q_J \\ P_J J Q_J \end{pmatrix} = \begin{pmatrix} E \\ J \end{pmatrix} \tag{61}$$

should exist. Here $J + \lambda E$ is taken to be equivalent to J while $r(J + \lambda E) = r(J) = l$.

For equation (60), we take into account the two decomposed operations of T' separately. We have

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \Lambda' \\ B_n \end{pmatrix} = \begin{pmatrix} \alpha(\Lambda' + \sigma B_n) \\ \gamma B_n \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \Lambda' + \sigma B_n \\ B_n \end{pmatrix} \tag{62}$$

with $\sigma = \frac{\beta}{\alpha}$. As shown in appendix B, there exist ILOs P_{B_n} and Q_{B_n} which satisfy

$$P_{B_n} \begin{pmatrix} \Lambda' + \lambda' B_n \\ B_n \end{pmatrix} Q_{B_n} = \begin{pmatrix} \Lambda' \\ B_n \end{pmatrix}. \tag{63}$$

And, furthermore we have

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \Lambda' \\ B_n \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \Lambda' \\ B_n + \frac{\alpha\lambda}{\gamma}\Lambda' \end{pmatrix}. \tag{64}$$

There also exist ILOs P'_{B_n} and Q'_{B_n} which satisfy (see appendix B)

$$P'_{B_n} \begin{pmatrix} \Lambda' \\ B_n + \lambda\Lambda' \end{pmatrix} Q'_{B_n} = \begin{pmatrix} \Lambda' \\ B_n \end{pmatrix}. \tag{65}$$

Thus, equation (60) becomes

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} P_B \Lambda' Q_B \\ P_B B_n Q_B \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \Lambda' \\ B_n \end{pmatrix}, \tag{66}$$

where $P_B = P'_{B_n} P_{B_n}$ and $Q_B = Q_{B_n} Q'_{B_n}$. By taking $P_0 = P_J \oplus \frac{1}{\alpha} P_B$ and $Q_0 = Q_J \oplus Q_B$, we have

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} P_0 \Lambda Q_0 \\ P_0 \Gamma Q_0 \end{pmatrix} &= \begin{pmatrix} \begin{pmatrix} E & 0 \\ 0 & \Lambda' \end{pmatrix} \\ \begin{pmatrix} J & 0 \\ 0 & \frac{\gamma}{\alpha} B_n \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} E & 0 \\ 0 & \Lambda' \end{pmatrix} \\ \kappa \begin{pmatrix} \frac{1}{\kappa} J & 0 \\ 0 & B_n \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \Lambda \\ \kappa \Gamma \end{pmatrix}. \end{aligned} \tag{67}$$

Then equation (54) can be expressed as

$$\begin{aligned} \begin{pmatrix} \Lambda \\ \Gamma' \end{pmatrix} &= P' \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \Lambda \\ \Gamma \end{pmatrix} Q' \\ &= P' P_0^{-1} \begin{pmatrix} \Lambda \\ \kappa \Gamma \end{pmatrix} Q_0^{-1} Q' \\ &= \begin{pmatrix} P'' \Lambda Q'' \\ P'' \kappa \Gamma Q'' \end{pmatrix}, \end{aligned} \tag{68}$$

which is just equation (55).

Equation (68) corresponds to two equations

$$\begin{cases} P'' \Lambda Q'' = \Lambda \\ P'' \kappa \Gamma Q'' = \Gamma'. \end{cases} \tag{69}$$

The equation $\Lambda = P'' \Lambda Q''$ requires P'' and Q'' taking the following form:

$$P'' = \begin{pmatrix} S & Y \\ 0 & p \end{pmatrix}; \quad Q'' = \begin{pmatrix} S^{-1} & 0 \\ X & q \end{pmatrix}, \tag{70}$$

respectively. Here, p and q are arbitrary complex numbers. Note that in order to guarantee P'' and Q'' to be non-singular matrices, p and q cannot be zero.

The canonical form of Γ' in $c_{N-1,l}$ gives further constraints on the patterns of P'' and Q'' . Of Γ' and Γ , in which B_3 takes the form of (44), they can be generally expressed as

$$\begin{pmatrix} \times & \times & \times & 0 & v_1 & 0 \\ \times & \times & \times & 0 & v_2 & 0 \\ \times & \times & \times & 0 & v_3 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{71}$$

In fact any element in set $c_{N-1,l}$ takes the same patten (71). From (70), P'' and Q'' take the forms of

$$P'' = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & y_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & y_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & y_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & y_4 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & y_5 \\ 0 & 0 & 0 & 0 & 0 & p \end{pmatrix}, \tag{72}$$

$$Q'' = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & 0 \\ x_1 & x_2 & x_3 & x_4 & x_5 & q \end{pmatrix},$$

where $S = \{a_{ij}\}_{5 \times 5} = \{b_{ij}\}_{5 \times 5}^{-1}$. From (71) and constraint $\Gamma' = P''\kappa\Gamma Q''$, P'' and Q'' read

$$P'' = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & y_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & y_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & y_3 \\ 0 & 0 & 0 & p\kappa & 0 & y_4 \\ a_{51} & a_{52} & a_{53} & a_{54} & 1/q\kappa & y_5 \\ 0 & 0 & 0 & 0 & 0 & p \end{pmatrix}, \tag{73}$$

$$Q'' = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & 0 & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & 0 & 0 \\ 0 & 0 & 0 & 1/p\kappa & 0 & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & q\kappa & 0 \\ x_1 & x_2 & x_3 & x_4 & x_5 & q \end{pmatrix}.$$

We note that if P'' and Q'' are invertible, then the upper-left submatrices $\{a_{ij}\}_{3 \times 3}$ and $\{b_{ij}\}_{3 \times 3}$ should also be invertible, due to the fact that any invertible matrix in the form $\begin{pmatrix} X & 0 \\ W & Y \end{pmatrix}$ has an inverse $\begin{pmatrix} X^{-1} & 0 \\ Z & Y^{-1} \end{pmatrix}$ with arbitrary matrices X and Y being also invertible. From (69) one may infer that in $P''\kappa\Gamma Q''$ only the block $\{a_{ij}\}_{3 \times 3}$ acts on a vector $v = \{v_i, i = 1, 2, 3\}$ in Γ of (71). Since no invertible operator can transform a nonzero vector into a null one, therefore $v = 0$ and $v \neq 0$ determine two ILO inequivalent cases for Γ . Thus, we see that if Γ' and Γ in $c_{N-1,l}$ satisfy equation (69), the identity of their

B_3 submatrices leads to the identity of their B_4 submatrices. Or in other words, for two matrices Γ and Γ' , if their submatrices B_3 are the same, while their B_4 submatrices are different, then they should be ILO inequivalent, like (47) and (48).

The above analysis for B_3 is applicable to other submatrices B_n with $n > 3$, e.g. the forms of B_4 and B_5 in Γ''_2 in equations (47)–(49). Generally every nonzero element in B_n will transform one column or one row of P'' and Q'' into a unit vector. In the end, under constraint (69), if two matrices Γ and Γ' have the same B_i , then they must possess the same B_{i+1} . According to theorem 1 we then have

$$P''\kappa \begin{pmatrix} J(\lambda_i) & 0 \\ 0 & B_n \end{pmatrix} Q'' = \begin{pmatrix} SJ(\kappa\lambda_i)S^{-1} & 0 \\ 0 & B_n \end{pmatrix}, \tag{74}$$

that is

$$\begin{bmatrix} \Lambda \\ \Gamma' \end{bmatrix} = \begin{bmatrix} \Lambda \\ \Gamma \end{bmatrix}. \tag{75}$$

(Note that there exists a special case in which the B matrix takes up the whole Γ and then (59) does not exist. The elements in the pivot of T' now can have zeros. In this situation, the only modification of the above proofing process is by adding two more ILOs P_t and Q_t which can reverse the flip of (Λ', B) induced by the zero elements in the pivot of T' , see appendix C for details.) □

3.3. Classification on set $C_{n,l}$ with $n = N - i$

The same procedure can be directly applied to the $C_{N-2,l}$ case, and so on. Taken here again the $N = 6$ case as an example, following the construction procedure in equation (44) it is easy to obtain the canonical form of (Λ, Γ) in $c_{6-2,l}$:

$$\Lambda = \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right), \tag{76}$$

$$\Gamma = \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right). \tag{77}$$

In this case, it is obvious that l cannot be smaller than 4. Rescale matrices (76) and (77) according to the partition lines we have

$$\Lambda = \begin{pmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{78}$$

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & E \\ E & 0 & 0 \end{pmatrix}. \tag{79}$$

Here Γ is just like B_3 in equation (44). Thus, the classification of the et $c_{N-i,l}$ with $i > 1$ can be reduced to the classification of $c_{N-1,l}$ in principle. From (44) and (77) we note that a proposition (inequality) of n, l in $c_{n,l}$ should exist, i.e.

$$2(N - n) \leq l \leq n, \tag{80}$$

when $n < N$. Equation (80) is a constraint on the rank of a matrix pair which represents the true entangled state of $2 \times N \times N$.

Now we have fully classified all the truly entangled classes of the $2 \times N \times N$ state. A truly entangled state of $2 \times N \times N$ must lie in one of the sets $C_{N-i,l}$ (or $C_{n,l}$). According to (80) we can obtain a restriction on the values of n :

$$\frac{2N}{3} \leq n < N \tag{81}$$

when $i > 0$, and from the arguments above (26) we know

$$n = N, \quad 0 < l < N \tag{82}$$

when $i = 0$. From those two theorems proved in this section we know that the mapping of $C_{N-i,l} \mapsto c_{N-i,l}$ determines all the true entanglement classes in $C_{N-i,l}$.

4. Examples

According to the above explanation, the classification of the entangled state $2 \times N \times N$ may be accomplished by repeatedly taking the above introduced procedures. To be more specific and for the reader's convenience, in the following we completely classify the $2 \times 2 \times 2$ and $2 \times 4 \times 4$ pure states by using this novel method as examples.

For $N = 2$, i.e. three-qubit states, there is only one inequivalent set $c_{N=2,l=1}$ (the case $c_{1,1}$ does not exist in the three-qubit true entanglement state due to proposition (80)). There are two inequivalent Jordan forms for 2×2 matrices with rank 1, and thus two inequivalent classes in $c_{2,1}$ which correspond to the GHZ and W states separately [7]:

$$\text{GHZ: } E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}, \tag{83}$$

$$\text{W: } E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{84}$$

For $N = 4$, from (80)–(82) the inequivalent sets are

$$c_{N,l} = c_{4,1}, c_{4,2}, c_{4,3} \tag{85}$$

$$c_{N-1,l} = c_{3,2}, c_{3,3}. \tag{86}$$

Due to (81), there are no $c_{N-i,l}$ sets in truly entangled states for $N = 4$ when $i \geq 2$. In the case of $c_{4,l}$ all inequivalent classes have the form of $\begin{pmatrix} E \\ J \end{pmatrix}$, where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{87}$$

Hence, we can distinguish them just by virtue of J 's pattern. There are two classes in the set $c_{4,1}$, i.e.

$$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad (88)$$

six classes in the set $c_{4,2}$,

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad (89)$$

and five classes in the set $c_{4,3}$,

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (90)$$

In the case of $c_{3,l}$, every class has the form of $\begin{pmatrix} \Lambda \\ \Gamma \end{pmatrix}$ where

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (91)$$

From theorem 2 in section 3.2 we can simply classify the set $c_{3,2}$ by the pattern of the Γ matrix. And, from the measure in constructing the matrices B_3 and B_4 in the same section, we find that there is one class in $c_{3,2}$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (92)$$

and two classes in $c_{3,3}$:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (93)$$

Altogether, there are 16 genuine entanglement classes in $2 \times 4 \times 4$ states, which agree with what was obtained in [12]. From (93) one can easily conclude that the permutation of the two four-dimensional partite is sorted into different classes, which was noted in [12].

In the above examples we enumerate various distinct classes of the $2 \times 2 \times 2$ and $2 \times 4 \times 4$ states, each with a representative state. In fact, there are still reducible parameters in the representative states, the eigenvalues of the Jordan form, e.g. the λ s in (89). These parameters can be sorted into two categories: one with only redundant parameters, which can be eliminated out of the state through ILOs; another possesses non-local parameters which cannot be eliminated through the ILOs and will exist in the entangled state as free parameters. The situation may become more complicated with the increase of the numbers of parameters. To get a deeper insight of the nature of those non-local parameters, there is still a lot of work that needs to be done.

5. Summary and conclusions

In conclusion, we put forward a novel method in classifying the entangled pure states of $2 \times N \times N$. A remarkable feature of our method being different from what existed in the literature is that it does not need to classify the lower dimension cases first [11, 12]. We find in practice that this method in classifying the $2 \times N \times N$ tri-partite entanglement state is quite straightforward. Since the software for Jordan decomposition is available, this new method can be applied to the classification of a given state via automatic computer calculation, which is very important as the partite dimension N tends to be large. Last but not the least, in this work a pictorial configuration of the entanglement states on the grids is proposed, which gives an intuitive demonstration for the entanglement classes, and is helpful in eliminating the remaining redundant parameters.

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Appendix A. The construction of the B matrix

$$P_1 \Gamma'_1 Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{A.1})$$

$$P_1 \Gamma'_2 Q_1 = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \gamma_{25} & \gamma_{26} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34} & \gamma_{35} & \gamma_{36} \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & \gamma_{44} & \gamma_{45} & \gamma_{46} \\ \gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & \gamma_{55} & \gamma_{56} \\ \gamma_{61} & \gamma_{62} & \gamma_{63} & \gamma_{64} & \gamma_{65} & \gamma_{66} \end{pmatrix}. \tag{A.2}$$

A direct observation on equation (A.2) tells that γ_{66} must be zero; otherwise one can find invertible operators P_x and Q_x which enable

$$P_x P_1 \Gamma'_1 Q_1 Q_x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.3}$$

$$P_x P_1 \Gamma'_2 Q_1 Q_x = \begin{pmatrix} \gamma_{11x} & \gamma_{12x} & \gamma_{13x} & \gamma_{14x} & \gamma_{15x} & 0 \\ \gamma_{21x} & \gamma_{22x} & \gamma_{23x} & \gamma_{24x} & \gamma_{25x} & 0 \\ \gamma_{31x} & \gamma_{32x} & \gamma_{33x} & \gamma_{34x} & \gamma_{35x} & 0 \\ \gamma_{41x} & \gamma_{42x} & \gamma_{43x} & \gamma_{44x} & \gamma_{45x} & 0 \\ \gamma_{51x} & \gamma_{52x} & \gamma_{53x} & \gamma_{54x} & \gamma_{55x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.4}$$

Given λ_i are the eigenvalues of the submatrix $\{\gamma_{ijx}\}_{5 \times 5}$ and $\lambda \neq -\lambda_i$, we will find that $r(P_x P_1 \Gamma'_2 Q_1 Q_x + \lambda P_x P_1 \Gamma'_1 Q_1 Q_x) = N > N - 1$. This contradicts the requirement that the maximum rank of $(t_{11} \Gamma_1 + t_{12} \Gamma_2)$ is $N - 1$.

Let $\gamma_{66} = 0$; equations (A.1) and (A.2) become

$$P_1 \Gamma'_1 Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.5}$$

$$P_1 \Gamma'_2 Q_1 = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \gamma_{25} & \gamma_{26} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34} & \gamma_{35} & \gamma_{36} \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & \gamma_{44} & \gamma_{45} & \gamma_{46} \\ \gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & \gamma_{55} & \gamma_{56} \\ \gamma_{61} & \gamma_{62} & \gamma_{63} & \gamma_{64} & \gamma_{65} & 0 \end{pmatrix}. \tag{A.6}$$

Since we are considering the true entanglement of $2 \times N \times N$ states, neither the last column nor the last row of the matrix in equation (A.6) can be completely zero. There exist ILOs P_2, Q_2 which satisfy the following equations:

$$P_2 P_1 \Gamma'_1 Q_1 Q_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.7}$$

$$P_2 P_1 \Gamma'_2 Q_1 Q_2 = \begin{pmatrix} \gamma'_{11} & \gamma'_{12} & \gamma'_{13} & \gamma'_{14} & \gamma'_{15} & 0 \\ \gamma'_{21} & \gamma'_{22} & \gamma'_{23} & \gamma'_{24} & \gamma'_{25} & 0 \\ \gamma'_{31} & \gamma'_{32} & \gamma'_{33} & \gamma'_{34} & \gamma'_{35} & 0 \\ \gamma'_{41} & \gamma'_{42} & \gamma'_{43} & \gamma'_{44} & \gamma'_{45} & 0 \\ \gamma'_{51} & \gamma'_{52} & \gamma'_{53} & \gamma'_{54} & \gamma'_{55} & 1 \\ \gamma'_{61} & \gamma'_{62} & \gamma'_{63} & \gamma'_{64} & \gamma'_{65} & 0 \end{pmatrix}. \tag{A.8}$$

An invertible operator

$$Q_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\gamma'_{51} & -\gamma'_{52} & -\gamma'_{53} & -\gamma'_{54} & -\gamma'_{55} & 1 \end{pmatrix} \tag{A.9}$$

makes

$$P_2 P_1 \Gamma'_1 Q_1 Q_2 Q_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.10}$$

$$P_2 P_1 \Gamma'_2 Q_1 Q_2 Q_3 = \begin{pmatrix} \gamma'_{11} & \gamma'_{12} & \gamma'_{13} & \gamma'_{14} & \gamma'_{15} & 0 \\ \gamma'_{21} & \gamma'_{22} & \gamma'_{23} & \gamma'_{24} & \gamma'_{25} & 0 \\ \gamma'_{31} & \gamma'_{32} & \gamma'_{33} & \gamma'_{34} & \gamma'_{35} & 0 \\ \gamma'_{41} & \gamma'_{42} & \gamma'_{43} & \gamma'_{44} & \gamma'_{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \gamma'_{61} & \gamma'_{62} & \gamma'_{63} & \gamma'_{64} & \gamma'_{65} & 0 \end{pmatrix}. \tag{A.11}$$

Here, γ'_{65} must be zero also; otherwise to keep the form of (A.10) unchanged the matrix in equation (A.11) can be transformed into

$$\begin{pmatrix} \gamma''_{11} & \gamma''_{12} & \gamma''_{13} & \gamma''_{14} & 0 & 0 \\ \gamma''_{21} & \gamma''_{22} & \gamma''_{23} & \gamma''_{24} & 0 & 0 \\ \gamma''_{31} & \gamma''_{32} & \gamma''_{33} & \gamma''_{34} & 0 & 0 \\ \gamma''_{41} & \gamma''_{42} & \gamma''_{43} & \gamma''_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{A.12}$$

Clearly this will lead to the same contradiction as γ_{66} does in (A.2) and (A.4). Thus, equation (A.11) becomes

$$P_2 P_1 \Gamma'_2 Q_1 Q_2 Q_3 = \begin{pmatrix} \gamma'_{11} & \gamma'_{12} & \gamma'_{13} & \gamma'_{14} & \gamma'_{15} & 0 \\ \gamma'_{21} & \gamma'_{22} & \gamma'_{23} & \gamma'_{24} & \gamma'_{25} & 0 \\ \gamma'_{31} & \gamma'_{32} & \gamma'_{33} & \gamma'_{34} & \gamma'_{35} & 0 \\ \gamma'_{41} & \gamma'_{42} & \gamma'_{43} & \gamma'_{44} & \gamma'_{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \gamma'_{61} & \gamma'_{62} & \gamma'_{63} & \gamma'_{64} & 0 & 0 \end{pmatrix}. \tag{A.13}$$

Applying the same procedure to the last row as we have performed to the last column, we have

$$\Lambda = P \Gamma'_1 Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.14}$$

$$\Gamma''_2 = P \Gamma'_2 Q = \left(\begin{array}{ccc|ccc} \times & \times & \times & 0 & c_{15} & 0 \\ \times & \times & \times & 0 & c_{25} & 0 \\ \times & \times & \times & 0 & c_{35} & 0 \\ \hline r_{31} & r_{32} & r_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right) = \begin{pmatrix} A & c \\ r & B_3 \end{pmatrix}, \tag{A.15}$$

where $P = \prod_i P_i$ and $Q = \prod_i Q_i$ are sequences of invertible operators P_i and Q_i , respectively.

Appendix B. The superpositions of Λ' and B_n

Equation (63) can be written in the following matrix equations:

$$\begin{cases} P_{B_n} (\Lambda'_n + \lambda B_n) Q_{B_n} = \Lambda'_n \\ P_{B_n} B_n Q_{B_n} = B_n. \end{cases} \tag{B.1}$$

Here, for the sake of clarity, we label Λ' with subscript n to indicate its dimension.

Below, we show inductively that the invertible matrices P_{B_n} and Q_{B_n} can always be constructed.

In the case $n = 1$, then $\Lambda' = 0$, $B_1 = 0$, the construction of invertible operators P_{B_1} , Q_{B_1} equation (B.1) is trivial.

In the case of $n = 2$, equation (B.1) becomes

$$\begin{cases} P_{B_2} \begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix} Q_{B_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ P_{B_2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Q_{B_2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{cases} \tag{B.2}$$

$P_{B_2} = E$ and Q_{B_2} of the form $\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}$ satisfy the above equations.

Suppose that for an arbitrary n , equation (B.1) is true; we show that $P_{B_{n+1}}$, $Q_{B_{n+1}}$ can also be constructed, satisfying

$$\begin{cases} P_{B_{n+1}} (\Lambda'_{n+1} + \lambda B_{n+1}) Q_{B_{n+1}} = \Lambda'_{n+1} \\ P_{B_{n+1}} B_{n+1} Q_{B_{n+1}} = B_{n+1}. \end{cases} \tag{B.3}$$

Here, either

$$B_{n+1} = \begin{pmatrix} 0 & r \\ 0 & B_n \end{pmatrix} \tag{B.4}$$

or

$$B_{n+1} = \begin{pmatrix} 0 & 0 \\ c & B_n \end{pmatrix}, \tag{B.5}$$

where $r(B_n) = n - 1$; the ranks of $\begin{pmatrix} r \\ B_n \end{pmatrix}$ and $(c \ B_n)$ are n . And

$$\Lambda'_{n+1} = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda'_n \end{pmatrix}. \tag{B.6}$$

In one example of $n = 5$, Λ' and B can be expressed as follows:

$$\Lambda'_{5+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{5+1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{B.7}$$

where $B_{5+1} = \begin{pmatrix} 0 & r \\ 0 & B_5 \end{pmatrix}$, $r = (0, 1, 0, 0, 0)$.

The operators $P_{B_{n+1}}$ and $Q_{B_{n+1}}$ can be constructed as follows:

$$P_{B_{n+1}} = \begin{pmatrix} 1 & X \\ 0 & E \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & P_{B_n} \end{pmatrix}, \tag{B.8}$$

$$Q_{B_{n+1}} = \begin{pmatrix} 1 & 0 \\ 0 & Q_{B_n} \end{pmatrix} \begin{pmatrix} 1 & -Y \\ 0 & E \end{pmatrix}, \tag{B.9}$$

where $Y = \lambda r Q_{B_n} + X \Lambda'_n$. Because the rank of $\begin{pmatrix} 0 & r \\ 0 & B_n \end{pmatrix}$ is unchanged under the invertible transformation, we can always find such an invertible operator $\begin{pmatrix} 1 & X \\ 0 & E \end{pmatrix}$ which satisfies

$$\begin{pmatrix} 1 & X \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 & r Q_{B_n} \\ 0 & B_n \end{pmatrix} = \begin{pmatrix} 0 & r \\ 0 & B_n \end{pmatrix}. \tag{B.10}$$

It is then easy to verify equation (B.3) using equations (B.8), (B.9). Note that $P_{B_{n+1}}$ and $Q_{B_{n+1}}$ constructed above correspond to the case of equation (B.4), and for the case of (B.5) the procedure is similar.

Along the same line, it can also be found that there exist such invertible operators P'_{B_n} , Q'_{B_n} that

$$\begin{cases} P'_{B_n} \Lambda'_n Q'_{B_n} = \Lambda'_n \\ P'_{B_n} (B_n + \lambda \Lambda'_n) Q'_{B_n} = B_n. \end{cases} \tag{B.11}$$

Appendix C. The flip of Λ'_n and B_n

Using equation (B.1) and equation (B.11), we show that there exist the following invertible matrices P_t, Q_t which flip Λ'_n and B_n , like

$$\begin{pmatrix} P_t \Lambda'_n Q_t \\ P_t B_n Q_t \end{pmatrix} = \begin{pmatrix} B_n \\ \Lambda' \end{pmatrix}. \quad (\text{C.1})$$

Provided

$$\begin{cases} P_{B_n}(\lambda)(\Lambda'_n + \lambda B_n) Q_{B_n}(\lambda) = \Lambda'_n \\ P_{B_n}(\lambda) B_n Q_{B_n}(\lambda) = B_n, \end{cases} \quad (\text{C.2})$$

and

$$\begin{cases} P'_{B_n}(\lambda) \Lambda'_n Q'_{B_n}(\lambda) = \Lambda'_n \\ P'_{B_n}(\lambda)(B_n + \lambda \Lambda'_n) Q'_{B_n}(\lambda) = B_n, \end{cases} \quad (\text{C.3})$$

it can be found that

$$P_t = P_{B_n}(\lambda) P'_{B_n} \left(-\frac{1}{\lambda} \right) P_{B_n}(\lambda), \quad Q_t = Q_{B_n}(\lambda) Q'_{B_n} \left(-\frac{1}{\lambda} \right) Q_{B_n}(\lambda) \quad (\text{C.4})$$

enable

$$\begin{pmatrix} P_t \Lambda'_n Q_t \\ P_t B_n Q_t \end{pmatrix} = \begin{pmatrix} -\lambda B_n \\ \frac{1}{\lambda} \Lambda'_n \end{pmatrix}, \quad (\text{C.5})$$

which is equivalent to (C.1) up to irrelevant coefficients.

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